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# EFFECIENCY IN THE SUPPLY OF REGIONAL PUBLIC GOODS THAT ARE SUBJECT TO CONGESTION

David A.Vardy
Queen's University

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

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## I. Introduction

The purpose of the present paper is to extend the theory of regional public goods to account for population congestion, or crowding, and then to introduce the possibility of congestion into boundary spillovers of benefits. The important papers by Williams (1966) and by Brainard and Dolbear (1967) treated the special case of spillover benefits where the benefits were rivalrous between regions but non-rivalrous, or purely public, within a region. In more recent papers Pauly (1970) and Vardy (1971a) extended the analysis for inter-regionally non-rivalrous benefits. This paper deals with congestion costs for regional public goods provided by provinces which belong to a federation and indicates how optimizing subsidies may be used by the federal government to correct for benefit spillovers.

The analysis is based on the assumption that for each province there is a social welfare function and that this welfare function is defined in terms of per capita consumption of private and public goods.[1] This formulation results from the variability of provincial population and follows from the argument that each provincial government performs as a club dedicated to the well-being of the average member.[2] Population is assumed fixed for the federation as a whole but citizens are free to move from one province to another in order to achieve maximum satisfaction. Preferences for provincial public goods are revealed through "voting-by-foot" where each province provides a differentiated public good.[3] Each province allows free migration and its public good output responds to changes in population.

The next section develops the theory of congested public goods and shows how congestion will affect the behaviour of a single province, which is exclusively concerned with the welfare of its residents. Section III introduces congestion into local public good benefits and into spillin benefits. Provincial and federal optimization conditions are also derived. Section IV concludes the paper.

## II. Variable Population and Congestion for a Single Province

In this section attention is focused upon a single province in which two goods, a regional public good and a private good, are provided. Private goods are provided by competitive private industry while the provincial government provides the public good and behaves like a competitive industry in so doing. The private good is assumed to be homogeneous in all member provinces of the federation and it serves as the numeraire good. The public good is differentiated between provinces and it is assumed that demand for the public good is known by the government, so that problems of preference revelation are ignored.

The province that is of interest is assumed in this section to be small in relation to the rest of the federation. This enables us to assume that the relevant economic variables in other provinces remain invariant to changes that are introduced in the analysis that follows.

Provincial population depends upon factor and product prices and upon the provincial public good. The quantity and quality of provincial public goods are important determinants of locational choice in this model. The basic model is now given by the following four equations:

$$(1) \qquad U = U(\tfrac{X}{N}, \zeta R) \ ,$$

$$(2) \qquad X = \bar{X}(L_X, T_X) \ ,$$

$$(3) \qquad R = \bar{R}(N - L_X, T_0 - T_X) \quad \text{and}$$

$$(4) \qquad N = \bar{N}(W - \tfrac{PR}{N}, \zeta R) \ .$$

Equation (1) is the social welfare function for the province with X as consumption of private goods while R is defined as consumption of public goods and N is provincial population. The coefficient $\zeta$ is equal to unity when R is a pure public good and $0 < \zeta < 1$ for a public good that is subject to congestion. In the analysis that follows, the following assumptions will be made: (a) $0 < \zeta < 1$, (b) $\zeta = \zeta(N)$ and $\zeta' \lesssim 0$. Equations (2) and (3) are the production functions for the private and the public good. In equation (2) $L_X$ is the number of workers employed in the private goods industry while $T_X$ is the corresponding amount of land used in that industry. In equation (3) the total number of workers is given by N and the total fixed amount of land is denoted by $T_0$. Population is introduced into the system through equation (4), which is the demand for residence function, where W

$$(W = \frac{\partial X}{\partial L_1} = P\frac{\partial R}{\partial N})$$

is the wage rate and P is the price of the public good in terms of the numeraire private good. The unit cost of the public good, P, is assumed for simplicity to remain constant. The term $P/N(=\pi)$ is the benefit tax price per person for the provincial public good and $W-\frac{PR}{N}$ is the wage rate net of tax. From one province to the other there are likely to be differences in the net wage, but it is assumed that the supply of labour offered by each worker is always the same. Subscripts will be used on variables to denote partial derivatives. The partial derivatives $\tilde{N}_1$ and $\tilde{N}_2$ are assumed to be positive.

It should be noted that the model does not require that all individuals be equal in order to employ equal benefit taxes. Differences in intra-marginal surplus may exist with respect to the public good in spite of the fact that all residents of the province have the same marginal valuation at the equilibrium level of public goods provision.

All mobile residents are assumed to be workers and all landlords are immobile. It is assumed for simplicity that there are relatively few landlords so that N will be used to denote the total (variable) population as well as the supply of workers.

The provincial welfare function is to be maximized subject to the constraints imposed by (2), (3) and (4). To derive the optimality conditions write the following Lagrangian:

$$(5)\quad L = U(\tfrac{X}{N},\zeta R) + \lambda_1[X-\bar{X}(L_X,T_X)] + \lambda_2[R-\bar{R}(N-L_X,T_0-T_X)] + \lambda_3[N-\bar{N}(P\bar{R}_1-P\tfrac{R}{N},\zeta R)]$$

From this expression the following first order conditions[4] may be derived:

$$(6)\quad \frac{\partial L}{\partial X} = \frac{U_1}{N} + \lambda_1 = 0,$$

$$(7)\quad \frac{\partial L}{\partial R} = \zeta U_2 + \lambda_2 + \lambda_3(\bar{N}_1\frac{P}{N}-\zeta\bar{N}_2) = 0,$$

$$(8)\quad \frac{\partial L}{\partial L_X} = -\lambda_1\bar{X}_1 + \lambda_2\bar{R}_1 = 0$$

$$(9)\quad \frac{\partial L}{\partial T_X} = -\lambda_1\bar{X}_2 + \lambda_2\bar{R}_2 = 0, \text{ and}$$

$$(10)\quad \frac{\partial L}{\partial N} = -\frac{U_1X}{N^2} + U_2\zeta' R - \lambda_2\bar{R}_1 + \lambda_3 - \lambda_3\bar{N}_1P\bar{R}_{11} - \lambda_3\bar{N}_1\frac{PR}{N^2} - \lambda_3\bar{N}_2\zeta' R = 0.$$

From (6), (7) and (8) it is possible to solve for $\lambda_1$, $\lambda_2$ and $\lambda_3$:

$$(11)\quad \lambda_1 = -\frac{U_1}{N},$$

$$(12)\quad \lambda_2 = -U_1\frac{P}{N}, \text{ and}$$

$$(13)\quad \lambda_3 = \frac{\zeta NU_2-U_1P}{\zeta N\bar{N}_2-\bar{N}_1P}$$

If these values are substituted into (10) then (14) is obtained.

$$(14) \quad -\frac{X}{N^2}U_1 + U_2\zeta' R + U_1\frac{P}{N}\bar{R}_1$$

$$+ \left(\frac{\zeta N U_2 - U_1 P}{\zeta N\bar{N}_2 - \bar{N}_1 P}\right)(1-\bar{N}_1 P\bar{R}_{11} - \bar{N}_1\frac{PR}{N^2} - \bar{N}_2\zeta' R) = 0.$$

If equation (14) is multiplied by $\frac{N}{U_1}$ then it may be simplified to (15), where $\frac{U_2}{U_1} = S$.

$$(15) \quad \left(\frac{X}{N} - \bar{X}_1 - NS\zeta' R\right)\left(\zeta\bar{N}_2 - \frac{\bar{N}_1}{N}P\right)$$

$$= (\zeta NS-P)(1-\bar{N}_1 P\bar{R}_{11} - \bar{N}_1\frac{PR}{N^2} - \bar{N}_2\zeta' R).$$

This may be further simplified so that it becomes:

$$(16) \quad \left(\frac{X}{N} - \bar{X}_1 - NS\zeta' R\right)\frac{dN}{dR} = \zeta NS-P$$

There are essentially two types of congestion possible in this model and both result from population pressure. The first is diminishing returns to land as the labour force increases. The second is the dissipation of benefits from the public good because people are competing for them. Both of these are reflected in the bracketed expression given in (16). The term $NS\zeta' R$ is the loss in welfare through congestion costs that results from an increase in population. The term $(\frac{X}{N} - \bar{X}_1)$ is more difficult to interpret since $\frac{X}{N}$ represents the average consumption of the private good while $\tilde{X}_l$ is the marginal product of an additional resident (worker). This means that $(\frac{X}{N} - \tilde{X}_l)$ reflects demand as well as production conditions so that it measures diminishing productivity rather imperfectly. The analysis suggests that when congestion of both types increases then public goods provision ought to be adjusted to correct for congestion, but that this will be true only when population is responsive to public output. If the bracketed term in equation (16) is positive and if $\frac{dN}{dR}$ is positive then welfare maximization requires that $\zeta NS > P$, i.e., "underproduction" of the public good. Increasing public goods congestion increases the value of the bracketed expression in (16), since $\zeta' < 0$ and hence $NS\zeta' R < 0$. Diminishing productivity of labour has the same effect since $\tilde{X}_l$ falls.

If $\frac{dN}{dR} = 0$ then the Samuelson condition, $\zeta NS=P$, would obtain and congestion costs would show up only through the appearance of $\zeta$ in that expression. Rewritten as $NS=P/\zeta$ it is clear that increasing population would then simply mean the manifestation of congestion costs through increasing costs for a unit of benefit. Clearly, the importance of congestion costs in this model, and their effect upon the optimality conditions will depend crucially upon the value of $\frac{dN}{dR}$.

## III. Congestion Costs for Benefit Spillovers

This section takes up the analysis of the previous section for a two-province federation, where there are reciprocal spillover benefits. For provincial public goods that are subject to congestion costs the conditions required to achieve joint welfare maximization are derived. Again it is assumed that each province produces a homogeneous private good and a single, differentiated public good.

Let the physical spillover of benefits from province 1 to province 2 be denoted as $\alpha_{12}R_1$, where $R_1$ is the level of public good provision in province 1. If the public good is purely public as between provinces 1 and 2 then $\alpha_{12} = \alpha_{11} = 1$ so that $\alpha_{12} + \alpha_{11} = 2$. If the public good is impurely public and incompletely non-rivalrous as between provinces then $0 < \alpha_{11} < 1$ and $0 < \alpha_{12} < 1$. If the provincial public good in province 2 is also subject to spillover then $\alpha_{21} > 0$ and if it is inter-provincially rivalrous as well then $0 < \alpha_{21} < 1$ and $0 < \alpha_{22} < 1$.

Consider the residents of province 1, who are assumed to exhibit some similarity in taste. Their preferred public good is denoted (and measured) by $R_1$. However, they receive $\alpha_{21}R_2$ units of public goods from province 2 through spillins. These spillins are imperfect substitutes for $R_1$ and are subject to a utility discount which will be denoted by $\beta_{21}$ per unit of $R_2$. This means that in province 1, spillins of $\alpha_{21}R_2$ are measured in terms of $\alpha_{21}\beta_{21}R_2$ units of equivalent

$R_1$. The term $\beta_{12}$ is similarly defined for province 2 and it enables $\alpha_{12}R_1$ units of spillin to be converted into $\alpha_{12}\beta_{12}R_1$ equivalent units of $R_2$. When these public goods are completely non-rivalrous then $\beta_{11} = \beta_{22} = 1$ by definition.

Let the possibility of congestion now be admitted and let $\beta_{11} = \beta_{22} = 1$ only for the own congested public good and allow $0<\beta_{11}<1$ and $0<\beta_{22}<1$ for congested public goods. Furthermore, let $\beta_{11} = \beta_{11}(N)$ and $\beta_{22} = \beta_{22}(M)$, where N is the population of province 1 and M the population of province 2. In the same spirit define $\beta_{12}$ and $\beta_{21}$ as conversion coefficients for spillins whose role it is to convert spillins into equivalent units of own uncongested public goods, and let $\beta_{12} = \beta_{12}(M)$ and $\beta_{21} = \beta_{21}(N)$. Finally, define the following:

$$\zeta_{11} = \alpha_{11}\beta_{11} = \zeta_{11}(N),$$

$$\zeta_{12} = \alpha_{12}\beta_{12} = \zeta_{12}(M),$$

$$\zeta_{22} = \alpha_{22}\beta_{22} = \zeta_{22}(M), \text{ and}$$

$$\zeta_{21} = \alpha_{21}\beta_{21} = \zeta_{21}(N).$$

The welfare functions for provinces 1 and 2 are given by (17) and (18):

$$(17) \quad U = U(\tfrac{X}{N}, \zeta_{11}R_1 + \zeta_{21}R_2) \quad \text{and}$$

$$(8) \quad W = W(\tfrac{Z}{M}, \zeta_{22}R_2 + \zeta_{12}R_1).$$

The formulation adopted here admits the possibility of rivalrous and non-rivalrous benefits, both intra- and inter-provincially. In (18) private goods consumption is denoted by Z.

The production functions are given as follows:

$$(19)\quad X = \bar{X}(N_X, T_X),$$

$$(20)\quad R_1 = \bar{R}_1(N-N_X, T_0-T_1),$$

$$(21)\quad Z = \bar{Z}(M_Z, K_Z),\ \text{and}$$

$$(22)\quad R_2 = \bar{R}_2(M-M_Z, K_0-K_Z).$$

Landlords are assumed here, as in the previous sections, to be immobile and small in number. They are assumed to own $T_0$ units of land in province 1 and $K_0$ units in province 2.

The demand for residence function in province 1 is given by

$$(23)\quad N = \bar{N}\left(\frac{\partial X}{\partial N_X} - \frac{P_1R_1}{N} - \frac{\partial Z}{\partial M_Z} + \frac{P_2R_2}{M},\ \zeta_{11}R_1+\zeta_{21}R_2,\ \zeta_{22}R_2+\zeta_{12}R_1\right).$$

As in section II it is assumed here that $P_1$ and $P_2$ are constant. The first argument of $\bar{N}$ is the difference between the wage rate net of tax in the two provinces. The second argument in $\bar{N}$ is the level of public good consumption (measured in units of uncongested $R_1$) available in province 1, while the third argument of $\tilde{N}$ refers to the corresponding availability of public goods in province 2. The final population equation is given by (24):

$$(24) \qquad N + M = N_0.$$

The welfare maximization problem for the federation may be presented in the form of the following Lagrangian expression:

$$(25) \quad L = U(\tfrac{X}{N}, \varsigma_{11}R_1+\varsigma_{21}R_2) + \lambda_1[W(\tfrac{Z}{M}, \varsigma_{22}R_2+\varsigma_{12}R_1)-W^0]$$

$$+ \lambda_2[X-\tilde{X}(N_X, T_X)] + \lambda_3[R_1-\tilde{R}_1(N-N_X, T_0-T_1)]$$

$$+ \lambda_4[Z-\tilde{Z}(M_Z, K_Z)] + \lambda_5[R_2-\tilde{R}_2(M-M_Z, K_0-K_Z)]$$

$$+ \lambda_6[N-\tilde{N}(\frac{\partial\tilde{X}}{\partial N_X} - \frac{P_1R_1}{N} - \frac{\partial Z}{\partial M_Z} + \frac{P_2R_2}{M},$$

$$\varsigma_{11}R_1+\varsigma_{21}R_2, \varsigma_{22}R_2 + \varsigma_{12}R_1)] + \lambda_7[N_0-N-M].$$

The first order conditions for welfare maximization are as follows:

$$(26) \quad \frac{\partial L}{\partial X} = \frac{U_1}{N} + \lambda_2 = 0,$$

$$(27) \quad \frac{\partial L}{\partial Z} = \lambda_1\frac{W_1}{M} + \lambda_4 = 0,$$

$$(28) \quad \frac{\partial L}{\partial R_1} = \varsigma_{11}U_2 + \lambda_1\varsigma_{12}W_2 + \lambda_3 + \lambda_6\tilde{N}_1\frac{P_1}{N} - \lambda_6\tilde{N}_2\varsigma_{11}$$

$$- \lambda_6\tilde{N}_3\varsigma_{12} = 0,$$

$$(29)\quad \frac{\partial L}{\partial R_2} = \zeta_{21}U_2 + \lambda_1\zeta_{22}W_2 + \lambda_5 - \lambda_6\bar{N}_1\frac{P_2}{M} - \lambda_6\bar{N}_2\zeta_{21}$$

$$- \lambda_6\bar{N}_3\zeta_{22} = 0,$$

$$(30)\quad \frac{\partial L}{\partial N_X} = - \lambda_2\frac{\partial\bar{X}}{\partial N_X} + \lambda_3\frac{\partial\bar{R}_1}{\partial N_X} = 0,$$

$$(31)\quad \frac{\partial L}{\partial T_X} = - \lambda_2\frac{\partial\bar{X}}{\partial T_X} + \lambda_3\frac{\partial\bar{R}_1}{\partial T_X} = 0,$$

$$(32)\quad \frac{\partial L}{\partial M_Z} = - \lambda_4\frac{\partial\bar{Z}}{\partial M_Z} + \lambda_5\frac{\partial\bar{R}_2}{\partial M_Z} = 0,$$

$$(33)\quad \frac{\partial L}{\partial K_Z} = - \lambda_4\frac{\partial\bar{Z}}{\partial K_Z} + \lambda_5\frac{\partial\bar{R}_2}{\partial K_Z} = 0,$$

$$(34)\quad \frac{\partial L}{\partial N} = - \frac{X}{N^2}U_1 + (\zeta'_{11}R_1+\zeta'_{21}R_2)U_2$$

$$- \lambda_3\frac{\partial\bar{R}_1}{\partial N} + \lambda_6 - \lambda_6\bar{N}_1\frac{P_1R_1}{N^2} - \lambda_6\bar{N}_1P\bar{R}_{11}$$

$$- \lambda_6\bar{N}_2\zeta'_{11}R_1 - \lambda_6\bar{N}_2\zeta'_{21}R_2 - \lambda_7 = 0,$$

$$(35)\quad \frac{\partial L}{\partial M} = - \lambda_1\frac{Z}{M^2}W_1 + \lambda_1(\zeta'_{22}R_2+\zeta'_{12}R_1)W_2$$

$$- \lambda_5\frac{\partial\bar{R}_2}{\partial M} + \lambda_6\bar{Z}_{11}\bar{N}_1 + \lambda_6\frac{P_2R_2}{M^2}\bar{N}_1$$

$$- \lambda_6\bar{N}_3\zeta'_{22}R_2 - \lambda_6\bar{N}_3\zeta'_{12}R_1 - \lambda_7 = 0.$$

Before solving these equations it will be useful to find the total differential of (23):

$$(36)\quad dN = \widetilde{N}_1(\widetilde{X}_{11}dN + \frac{P_1R_1}{N^2}dN - \frac{P_1}{N}dR_1 + \widetilde{Z}_{11}dN$$

$$+ \frac{P_2R_2}{M^2}dN + \frac{P_2}{M}dR_2)$$

$$+ \widetilde{N}_2(\zeta'_{11}R_1dN + \zeta'_{21}R_2dN + \zeta_{11}dR_1 + \zeta_{21}dR_2)$$

$$+ \widetilde{N}_3(\zeta_{22}dR_2 + \zeta_{12}dR_1 - \zeta'_{22}R_2dN - \zeta'_{12}R_1dN), \text{ or}$$

$$(37)\quad dN(1-\widetilde{N}_1\widetilde{X}_{11} - \frac{P_1R_1}{N^2}\widetilde{N}_1-\widetilde{N}_1\widetilde{Z}_{11}-\widetilde{N}_1\frac{P_2R_2}{M^2}$$

$$- \widetilde{N}_2\zeta'_{11}R_1-\widetilde{N}_2\zeta'_{21}R_2+\widetilde{N}_3\zeta'_{22}R_2+\widetilde{N}_3\zeta'_{12}R_1)$$

$$= dR_1(\widetilde{N}_2\zeta_{11}+\widetilde{N}_3\zeta_{12}-\widetilde{N}_1\frac{P_1}{N}) + dR_2(\widetilde{N}_1\frac{P_2}{M} + \widetilde{N}_2\zeta_{21}+\widetilde{N}_3\zeta_{22}).$$

This may be rewritten as

$$(38)\quad dN = \frac{H_1}{H_3}dR_1 + \frac{H_2}{H_3}dR_2.$$

Substituting $H_1$ and $H_2$ into (28) and (29) the following equations are obtained:

$$(39)\quad \zeta_{11}U_2+\lambda_1\zeta_{12}W_2+\lambda_3-H_1\lambda_6 = 0, \text{ and}$$

$$(40)\quad \zeta_{21}U_2+\lambda_1\zeta_{22}W_2+\lambda_5-H_2\lambda_6 = 0.$$

From (34) and (35) it is possible to derive (41):

$$(41)\quad -\frac{X}{N^2}U_1+(\zeta'_{11}R_1+\zeta'_{21}R_2)U_2-\lambda_3\frac{\partial\bar{R}_1}{\partial N}+\lambda_6-\lambda_6\bar{N}_1\frac{P_1R_1}{N^2}-\lambda_6\bar{N}_1\bar{X}_{11}$$

$$-\lambda_6\bar{N}_2\zeta'_{11}R_1-\lambda_6\bar{N}_2\zeta'_{21}R_2$$

$$=-\lambda_1\frac{Z}{M^2}W_1+\lambda_1(\zeta'_{22}R_2+\zeta'_{12}R_1)W_2-\lambda_5\frac{\partial\bar{R}}{\partial M}+\lambda_6\bar{Z}_{11}\bar{N}_1+\lambda_6\frac{P_2R_2}{M^2}\bar{N}_1$$

$$-\lambda_6\bar{N}_3\zeta'_{22}R_2-\lambda_6\bar{N}_3\zeta'_{12}R_1,\quad\text{or}$$

$$(42)\quad \frac{X}{N^2}U_1-(\zeta'_{11}R_1+\zeta'_{21}R_2)U_2-\lambda_1\frac{Z}{M^2}W_1+\lambda_1(\zeta'_{22}R_2+\zeta'_{12}R_1)W_2$$

$$+\lambda_3\frac{\partial\bar{R}_1}{\partial N}-\lambda_5\frac{\partial\bar{R}_2}{\partial M}=H_3\lambda_6$$

From (30) and (31) it is easy to derive $\lambda_3 = P_1\lambda_2$ and $\lambda_5 = P_2\lambda_4$. Now rewrite (42) as

$$(43)\quad \frac{U_1}{N}\frac{A}{H_3}-\lambda_1\frac{W_1}{M}\frac{B}{H_3}-\frac{C}{H_3}U_2+\lambda_1\frac{D}{H_3}W_2=\lambda_6,$$

where $A = \frac{X}{N} - \frac{\partial\bar{X}}{\partial N_X}$,

$B = \frac{Z}{M} - \frac{\partial\bar{Z}}{\partial M_Z}$,

$C = \zeta'_{11}R_1+\zeta'_{21}R_2$, and

$D = \zeta'_{22}R_2+\zeta'_{12}R_1$.

Substitute (43) into (39) and (40) to get (44) and (45).

(44) $$\zeta_{11}U_2+\lambda_1\zeta_{12}W_2 = P_1\frac{U_1}{N} + [\frac{U_1}{N}A-\lambda_1\frac{W_1}{M}B-CU_2+\lambda_1 DW_2]\frac{dN}{dR_1},$$

and

(45) $$\zeta_{21}U_2+\lambda_1\zeta_{22}W_2 = P_2\frac{W_1}{M}\lambda_1 + [\frac{U_1}{N}A-\lambda_1\frac{W_1}{M}B-CU_2+\lambda_1 DW_2]\frac{dN}{dR_2}.$$

By the assumption that the private good is the numeraire then $\lambda_2 = \lambda_4$ so that $\lambda_1 = \frac{U_1}{N}\frac{M}{W_1}$. This enables simplification of (44) and (45) to get

(46) $$\zeta_{11}NS_1+\zeta_{12}MS_2-P_1$$

$$= [A-B-(\zeta'_{11}R_1+\zeta'_{21}R_2)NS_1+(\zeta'_{12}R_1+\zeta'_{22}R_2)MS_2]\frac{dN}{dR_1}$$

and

(47) $$\zeta_{21}NS_1+\zeta_{22}MS_2-P_2$$

$$= [A-B-(\zeta'_{11}R_1+\zeta'_{21}R_2)NS_1+(\zeta'_{12}R_1+\zeta'_{22}R_2)MS_2)]\frac{dN}{dR_2}$$

The terms $S_1$ and $S_2$ are the marginal rates of substitution of private for public goods in provinces 1 and 2.

The two equations (45) and (46) are the first order efficiency conditions for the production of the two provincial public goods. If $\frac{dN}{dR_1} = \frac{dN}{dR_2} = 0$ then these equations replicate the familiar Samuelson conditions. In this situation independent adjustment would lead to equilibrium situations at which

$$\zeta_{11}NS_1 = P_1 \quad \text{and} \quad \zeta_{22}MS_2 = P_2 .$$

It is well known that optimality would then require some expansion of public output by each province and that this may be achieved by bargaining between the provinces or through federal subsidies.

Let it now be assumed that $\frac{dN}{dR_1} > 0$ and $\frac{dN}{dR_2} < 0$ and remember that $\zeta'_{ij} < 0$. An increase in public good output in province 1 will attract more people and create more congestion in that province. However, it will relieve congestion in province 2 so that the sign of $\zeta'_{11}R_1NS_1 - \zeta'_{12}R_1MS_2$ will reflect both of these influences. In the absence of bargaining or federal intervention the two provinces may achieve equilibrium positions at which

$$(47) \quad \zeta_{11}NS_1 - P_1 = (A - \zeta'_{11}R_1NS_1)\frac{dN}{dR_1}, \text{ and}$$

$$(48) \quad \zeta_{22}MS_2 - P_2 = (\zeta'_{22}R_2MS_2 - B)\frac{dN}{dR_2}.$$

This implies self-interest welfare maximization on the part of each province. On the basis of this behavioural assumption it is possible to derive the following optimizing federal subsidies:

$$(49) \quad V_1 = \zeta_{12}MS_2 + [B + \zeta'_{21}R_2NS_1 - (\zeta'_{12}R_1 + \zeta'_{22}R_2)MS_2]\frac{dN}{dR_1}.$$

and

$$(50) \quad V_2 = \zeta_{21}NS_1 - [A - (\zeta'_{11}R_1 + \zeta'_{21}R_2)NS_1 + \zeta'_{12}R_1MS_2]\frac{dN}{dR_2}.$$

In equations (49) and (50) the term $V_1$ is the per unit subsidy to province 1 and $V_2$ is the corresponding subsidy to province 2. These subsidies are calculated to ensure that the self-interested behaviour of the two provinces will lead to the optimal level of public output for the federation. These subsidies are non-linearly related to public output, as is evident from the variable coefficients of $R_1$ and $R_2$. The subsidy level varies inversely with own congestion and directly with congestion in the other province.

## IV. Conclusion

This paper introduces congestion costs for a model of regional public goods with benefit spillovers and variable population. The basic model requires that public goods exercise a degree of locational pull or attraction upon people. In Section II the analysis was presented for a single province and in Section III for two regions with benefit spillovers. When congestion is present then the output of the public good will be adjusted to achieve a more optimal size of population. A province has no direct control over its population in the models presented above and its only indirect control is through the provincial public good. If population equals or exceeds the welfare maximizing population level and if potential residents would be attracted by an increase in the output of the provincial public good (with its associated costs) then the public good should be "undersupplied". This means that the optimal solution is one for which summed marginal valuations for residents ought to exceed the price of the public good.

When federal subsidies are employed to achieve optimality they must be designed to achieve a welfare-maximizing distribution of population in the federation through an appropriate output of public goods. This is true with or without benefit spillovers. Rewrite (46) and (47) for the case where $\zeta_{12} = \zeta_{21} = 0$:

$$(51) \quad \zeta_{11}NS_1 - P_1 = [A - B - \zeta'_{11}R_1NS_1 + \zeta'_{22}R_2MS_2]\frac{dN}{dR_1},$$

and

$$(52) \quad \zeta_{22}MS_2 - P_2 = [A - B - \zeta'_{11}R_1NS_1 + \zeta'_{22}R_2MS_2]\frac{dN}{dR_2}.$$

Equations (51) and (52) indicate that even in this case there is likely to be an externality problem, as evidenced, for example, by the presence of $(\zeta'_{22}R_2MS_2 - B)$ in equation (51). In order to achieve maximum welfare for the federation as a whole some adjustment in local government behaviour is required. At the heart of the problem is the optimal inter-regional allocation of people in a world characterized by diminishing returns to fixed factors and congested public goods.

## FOOTNOTES

1. See Vardy (1971b) for the development of the basic model for uncongested public goods with endogenous population.

2. For the seminal contribution to the theory of clubs see Buchanan (1965).

3. See Tiebout (1956) for the seminal contribution to the migrant theory of local public goods and product differentiation.

4. It is assumed that the second order conditions for a welfare maximum are satisfied.

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